

Literally Changing the Point of View

*Matthias Kawski**

kawski@asu.edu

School of Mathematical and Statistical Sciences

Arizona State University

Tempe, Arizona 85287, USA

Abstract

For many years computer graphics has been used in calculus-level classrooms to visualize diverse mathematical objects. In the two dimensional case, this usually encompasses reproducing images that are familiar from centuries of masterful art practiced in books and on blackboards. Unfortunately, in the three dimensional case, many uses are of a disappointingly similar character, instead of pouncing on the opportunity to get new and deeper insights that previously were beyond reach. We demonstrate how three dimensional visualization can link together formerly disparately treated topics, cultivate the central mathematical ideas of lifting and projecting, and foster the habit of exploring new points of view. This article gives three innovative examples that do the latter. It challenges the reader to reconsider all situations that involve more than two variables: maybe there is a better three-dimensional view that is not part of the tradition just because one could not do it at the time. The examples discussed address constrained optimization problems, compositions of functions, and the product rule of differentiation.

1 Introduction

For millennia, mathematicians have perfected the art of providing two-dimensional graphical illustrations to support abstract reasoning. We are so accustomed to many such classical pictures that we rarely ask whether there are better alternatives. In the past there were not many choices as pictures of 3D objects are hard to create by hand. By putting 3D graphics at anyone's fingertips recent computer technology has obviated the need to make everything two-dimensional. In addition to the aesthetic aspects, the full 3D view often gives considerably more insight.

This article gives three elementary examples to illustrate this point. The first connects, in a single picture, the common views of elementary optimization techniques of single and multi-variable calculus courses. The two others are selected from the many first semester calculus and precalculus topics that truly involve three variables. We provide 3D views of the product rule and the composition of

*This work was partially supported by NSF funding through the Foundation Coalition under Cooperative Agreement EEC 92-21460 and the grant DMS 09-08204.

two functions. The readers are invited to find analogous pictures for the chain rule, related rates, and integration by parts. We propose:

Challenge: Rethink whether each topic that involves three variables would benefit from a 3D graphical representation.

The pictures presented here cannot do justice to this paradigm. For the most captivating experience, the student or the reader should rotate the objects himself or herself by simply clicking and dragging. To facilitate this we include sample commands for MAPLE. But with almost any 3D graphing program similar images are created just as readily. In order to focus on the central idea, the following examples employ traditional notation that is familiar from calculus classes. This includes some common abuse of notation, and we also omit technical discussions of well-known special cases.

This material is an outgrowth of a presentation at the *Third Conference on the Teaching of Calculus* (Ann Arbor, Michigan, 1994). In the intervening years, the use of computers for visualizing calculus concepts has become more routine – but all too often new technology is only used to reproduce the old images. Rather than just doing with technology what one used to do by hand, the real challenge is to creatively utilize new technologies to do the things that we never could do before.

2 Constrained optimization

A core topic in calculus is to optimize a function $f(x, y)$ of two variables subject to a constraint $g(x, y) = 0$. While other examples provide better pictures, for the sake of argument, consider the most basic problem of finding the rectangle of maximal area A among all rectangles with perimeter $L = 12$.

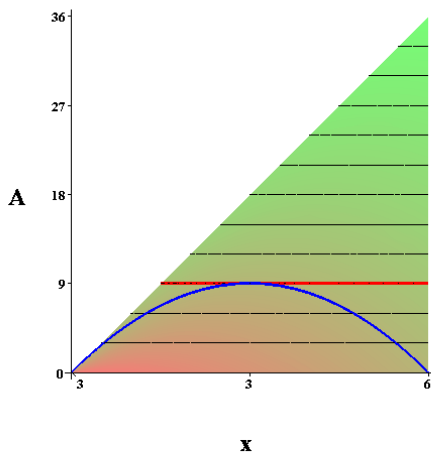


Fig. 1.a: One side view:
 $A = x(6 - x)$. Solve $\frac{dA}{dx} = 0$

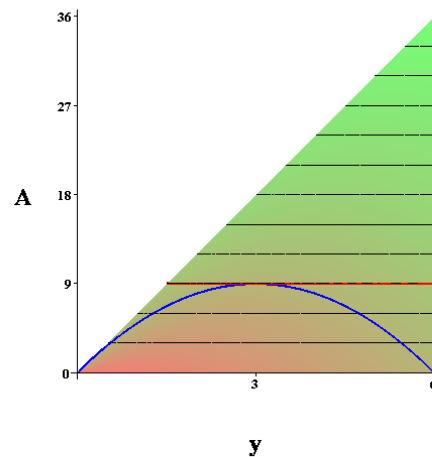


Fig. 1.b: Other side view:
 $A = (6 - y)y$. Solve $\frac{dA}{dy} = 0$

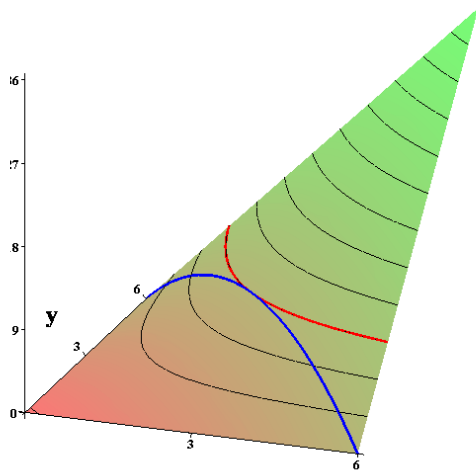


Fig. 1.c: The full 3D view:
 $A = xy, x + y = 6.$

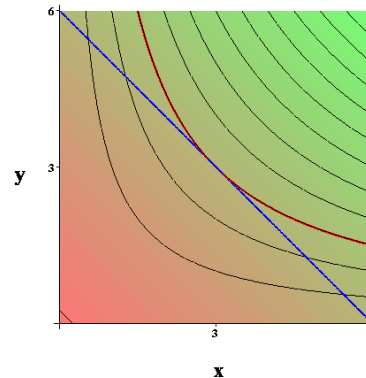


Fig. 1.d: Top view:
 Solve $\vec{\nabla} f + \lambda \vec{\nabla} g = 0, g = 0$

In a typical single variable course, denoting by x and y the lengths of the sides of the rectangle, the standard procedure is to solve the constraint for one variable, e.g. $y = h(x)$, substitute for this in the objective function, and calculate the critical points of an unconstrained problem, solving $0 = \frac{d}{dx} f(x, h(x))$. Here, $h(x) = 6 - x$, $A = xy = x(6 - x)$, and $\frac{dA}{dx} = 0$ iff $x = y = 3$. Of course, one may as well switch the role of the two variables. In either case one arrives a side view as shown in Figures 1.a. and 1.b.

A hybrid approach via implicit differentiation is seen less frequently: Equate $g_x(x, y) + g_y(x, y)y' = 0$ and $f_x(x, y) + f_y(x, y)y' = 0$ by identifying y' in both equations and using the constraint. In practical examples this is often the most efficient method, and is the one that first comes to mind when confronted with the *top-view* shown in Figure 1.d. A slightly more formal version of this approach uses only differentials and algebra. Solve $g_x(x, y) dx + g_y(x, y) dy = 0$, $f_x(x, y) dx + f_y(x, y) dy = 0$, and $g(x, y) = 0$. For an article advocating differentials see [1], but we note that differentials as *dual objects* are notoriously harder to correctly visualize.

In a multi-variable calculus course the problem is revisited using Lagrange multipliers: Form the function $F(\lambda, x, y) = f(x, y) + \lambda g(x, y)$ and solve the system $F_x = F_y = F_\lambda = 0$. In the example $F(\lambda, x, y) = xy + \lambda(12 - 2x - 2y)$, and the critical point $x = y = 3$ is obtained from the three simultaneous equations $y - 2\lambda = x - 2\lambda = 12 - 2x - 2y = 0$. Few students make the connection between the different approaches. It is not enough if the teacher *verbally* asserts: “These are *simply different points of view of the same problem.*” Our pictures prove this claim, taken literally. Rotating the 3-D computer graphics comes very close to taking the problem into one’s own hands, and physically rotating it to take different points of view.

Actual teaching experiences have been extremely good: Before introducing Lagrange multipliers we briefly review the single variable calculus technique, using a simple example. Then we switch on the projector to show the standard side view. This is a rare instance where it is worth it to prepare the graphics before class (so that it does not give away the punch-line). A brief discussion follows why this can’t be the full picture (the graph of a function of two variables is a surface). Then click on the graphics and rotate it into a general position as in Figure 1.c., exposing the full 3D object *behind*

the earlier seen picture. By now, well into the multi-variable course, the students are used to rotating graphs, and after further rotation quickly obtain the other side view, which corresponds to eliminating x and considering $A = y(6 - y)$. The following fundamental observation deserves some reflection:

Fact: The graphical analogue of (the algebraic operation of) eliminating a variable from a system of equations is rotating a graph until the corresponding axis disappears from view (as it becomes the line of sight).

If a system cannot be solved uniquely for a variable, the corresponding graphical object violates some liberal variation of the vertical or horizontal line test.

After looking at the box from one side and then from another side, it is natural to also look at it from the top. The students do this all the time, in particular, when introduced to multi-variable calculus via the extensive use of contour diagrams that is suggested by many client disciplines, compare the modern calculus text [2].

It is important that the plot style is chosen to show contours. The usual picture for introducing Lagrange multipliers arises: The maximum occurs at the point where the graph of the constraint is tangent to the level curves of the objective function. It is a worthwhile exercise to ask the students to explicitly write out the details behind this argument. Many students find it challenging to formulate the contrapositive: The maximum cannot occur at a point where the graph of the constraint crosses the level curves transversally. The most direct algebraic characterization of this point is in terms of the equations obtained above via implicit differentiation, liberally writing $y' = y'$. The usual algebraic algorithm in terms of Lagrange multipliers, that is suited for higher dimensional cases, this characterization is rephrased in terms of collinear gradients $\vec{\nabla} f$ and $\vec{\nabla} g$.

The complete picture actually involves the graph of the function $g(x, y)$ as well. Note that in practical problems f and g often have different dimensions (here one is a length and the other is an area). Consequently the full picture should have four axes. A nice way is to encode the function $g(x, y)$ using color. In MAPLE use the option `color=g(x, y)` in the `plot3d` command.

Alternatively, ignoring this remark, one may overlay two surfaces. This is helpful when discussing the relation to the dual problem, i.e. minimizing g subject to $f = \text{const}$. To emphasize that in general f and g do not take any commensurate values, it is a good idea to space the surfaces vertically far apart from each other. The key ingredient to make the method work, of course, is that the gradients both lie in the same xy -plane on the bottom, together with all the (projected) contours. Whereas the previous picture showed one fixed curve together with one family of contours, now one has two families of contours. Correspondingly, this leads one to correctly characterize the critical point by the linear dependence of the gradients, i.e., points (x, y) for which there exists $\lambda_0, \lambda_1 \in \mathbb{R}$ such that $\lambda_0 \vec{\nabla} f + \lambda_1 \vec{\nabla} g = 0$.

More abstract approaches to develop the Lagrange multiplier method are appropriate in advanced settings, compare [3]. But at an elementary level this usual picture is more redeeming. Moreover, it offers a hands-on approach to deal with the abnormal case, when the extremum occurs at a point where the gradient of g vanishes. It is hard for anyone who has learned the technique from the above pictures to miss the abnormal case. Indeed, correctly, the Lagrangian should be written as $F(\lambda_0, \lambda, x, y) = \lambda_0 f(x, y) + \lambda g(x, y)$. Yet most traditional calculus textbooks on the market in the United States are quite careless, tacitly assuming that one can choose a nonzero $\lambda_0 = 1$ with technical remarks about the other case nicely put a places where most readers do not really notice them. Thus it

is of little surprise that student who learnt from such books later make the common mistake of missing abnormal minimizers in advanced settings. We believe that by firmly basing the method of Lagrange multipliers on this graphical approach such mistakes can be largely avoided.

Extension

A closely related but more familiar picture arises in an important special case of the chain rule. Let $\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^2$ describe a planar curve, $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ a scalar function of two variables. Set $F = f \circ \vec{r}$. Then $F' = (\vec{\nabla} f \circ \vec{r}) \cdot \vec{v}$ where $\vec{v} = \vec{r}'$ denotes the velocity vector. While the picture is essentially the same, the interpretations, e.g. *walking over a hill*, or *temperature as a function of location as a function of time*, are different. Good choices for the *hill* are given by bounded rational functions. A nice picture results from $f(x, y) = \frac{1}{1+x^2+y^2}$ and $(x(t), y(t)) = (t, 1 - t^2)$.

3 The product rule

The traditional picture to support the algebraic manipulations for the product rule is the front view shown in Figure 2.a. While many standard textbooks do not have any illustrations for this at all, there are several that at least show this picture, e.g. [2, 5]. This is a prime example for the price traditional mathematics pays for reduction to two-dimensional illustrations. This is perfectly sound from an algebraic perspective working with differentials $d(u \cdot v) = du \cdot v + u \cdot dv$, compare e.g. [1]. But most first-year calculus students feel less comfortable with differentials than with differential quotients. In our experience, even though the independent variable occurs nowhere in the diagram, very few students dare to ask for the relation of the picture to the analytic expressions in the product rule $\frac{d}{dx}(f(x) \cdot g(x)) = \frac{df}{dx}(x) \cdot g(x) + f(x) \cdot \frac{dg}{dx}(x)$.

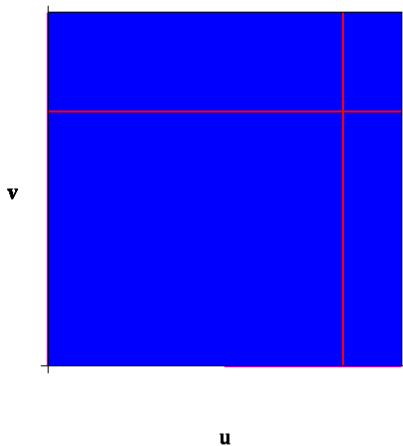


Fig. 2.a: The traditional front-view

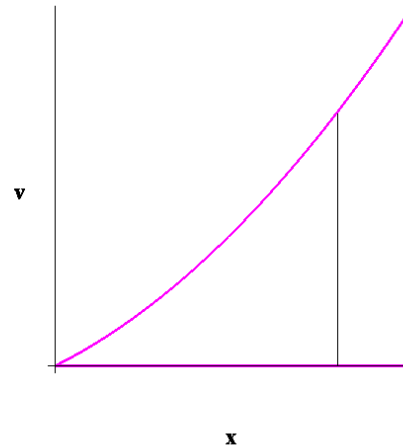


Fig. 2.b: The side view: $dv = g'(x) dx$

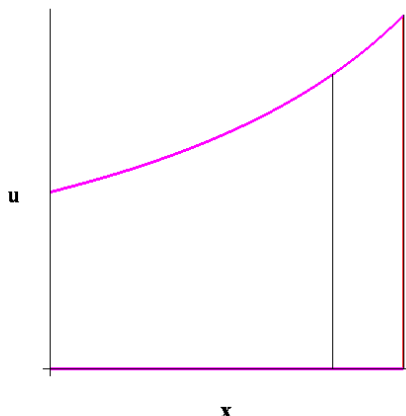


Fig. 2.c: The top view: $du = f'(x) dx$.

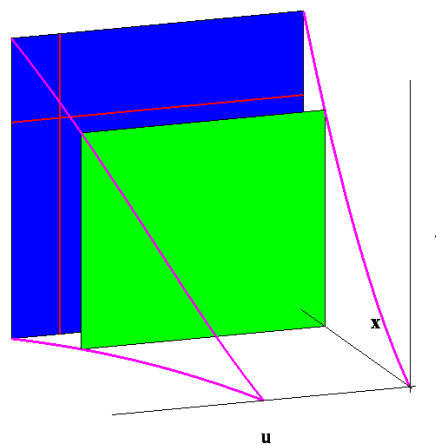


Fig. 2.d: The full 3D view

The 3D resolution is obvious: Three variables (the independent x , and the dependent u and v) deserve three axes. The side and top view shown in Figures 2.b and 2.c. show the graphs of u and v as functions of x . Students know where to find dx , du and $\frac{du}{dx}$, or dx , dv and $\frac{dv}{dx}$ in these familiar views. Click and drag to rotate into full 3D view shown in Figure 2.d. Carefully keeping track of all quantities, slowly continue to rotate until the x direction disappears from view. Alternatively, again play the game of starting with the uv -view as seen in the text book. Ask the class: “Where is x ?”, then click and drag to “tell the story behind the flat picture”.

Extension

Other occasions where 3D pictures arise from one independent and two dependent variables are *related rates* and *implicit equations*. The proverbial ladder leans against a wall and slides down. A two dimensional animation is an exciting starting point. Overlaying the frames on top of each other corresponds to a third (time-) axis pointing towards the observer. A critical step is to go from the curves $(t, x(t), 0)$ and $(t, 0, y(t))$ traced by the foot and bottom, to the mathematically more interesting curve $(t, x(t), y(t))$.

Three-dimensional graphs are more commonly seen when graphing implicit equations using parameterizations. The standard picture arises from the circle $x^2 + y^2 = 1$ parameterized via $x = \cos t$ and $y = \sin t$. The 3D picture is the helix $(t, \cos t, \sin t)$. Reparameterizations as $(t, \cos \sqrt{t}, \sin \sqrt{t})$ or $(t, \cos t^2, \sin t^2)$ already illustrate the usefulness of the full 3D view (helices with nonconstant pitches), and of the side views (e.g. illustrating $\frac{dx}{dt}$ and $\frac{dy}{dt}$ graphically).

It is only a small step from such parameterized curves to autonomous second order differential equations: The *side view* shows the solution $x = \sin t$ of the initial value problem $\ddot{x} + x = 0$, $x(0) = 0$, $\dot{x}(0) = 1$. Some like to overlay the other side view showing $\dot{x} = \cos t$, but this may actually hinder understanding. To graphically represent the differential equation, we first rewrite it as a first order system $\dot{x} = y$, $\dot{y} = -x$, and then draw the matching field plot in the xy -plane. The 3D view of the solution curve is the same helix as above. To avoid cluttering the picture we prefer to draw the arrows for the vector field only in the $t = 0$ plane. A far more advanced task is to get a similar picture for nonautonomous equations. The key is to draw the field on only a few planes

$t = \text{const}$ (that are spaced sufficiently far apart), and to code them with color. (Other uses for this picture are demonstrations of controlled or parameter-dependent differential equations; this will be the subject of a forthcoming article). A prime example for an intensive use of closely related graphical representations is [4]; however, there, in place of the 3D picture, the corresponding projections onto the coordinate planes are drawn in one combined diagram.

4 Composition of functions

In our experience most undergraduate students have a very limited understanding of functions as operators, and confuse the output $y = f(x)$ of a function f (given the input x) with the function itself. Many textbooks even call y a function of x . While convenient at some times, it causes much trouble at others. With the traditional emphasis on algebraic manipulations, it comes as no surprise when students interpret symbols like $y = \sin x^2$ as a juxtaposition of the symbols \sin and x^2 , and even mistake it for a product. These are often the same students who say “f x” in place of “f of/at x”, and who continue writing $y = fx$. Writing the derivative as $y' = \cos 2x$ and the complete inability of finding the graph of the composition of two functions that are presented graphically are almost necessary consequences. (E.g. provide the graphs of $f(u) = |1 - |u - 1||$ and $g(x) = |3 - x|$, and ask for the graph of the composition $f \circ g$.) Few students have any idea how to even get started, without a graphing calculator, to describe the graphs of $F(x) = e^{\sin x}$ and $G(x) = \sin e^x$.

A fundamental task for precalculus students is to compose two given tables of function values (preparing a table of the values of the composition). Some texts support the reasoning by drawing three parallel (!) axes representing the variables x , u and y . The usefulness of this representation is primarily limited to compositions of monotone functions and to local considerations (local linearity, chain-rule).

For a hands-on 3D graphical exercise put the graph of $u = g(x)$ flat on the table. Starting at $(x_0, 0, 0)$, go to the graph of g at $(x_0, g(x_0), 0)$ and over to the u -axis, i.e. to $(0, u_0, 0) = (0, g(x_0), 0)$.

Using a second piece of cardboard adjoin a vertical graph of $y = f(u)$, with the two u axes agreeing. Go from $(0, u_0, 0)$ to $(0, u_0, f(u_0))$ on the second graph. This laborious hands-on experience with cardboard is much accelerated by 3D computer graphics.

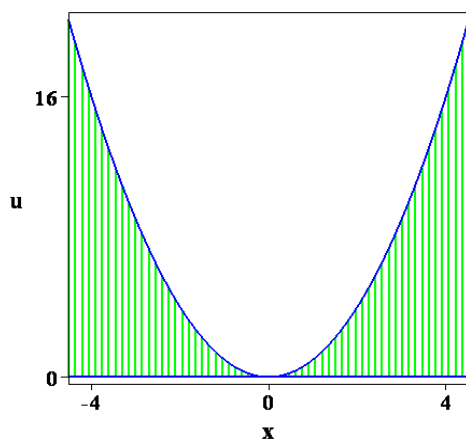


Fig. 3.b: The top view: $u = g(x)$

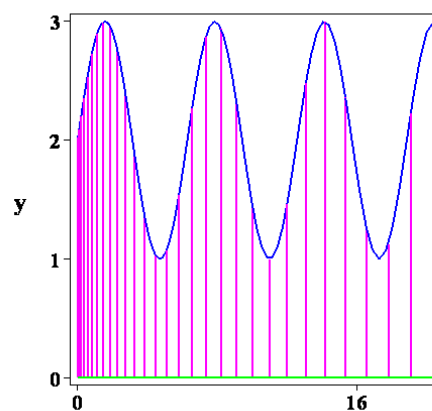


Fig. 3.b: The side view $y = f(u)$

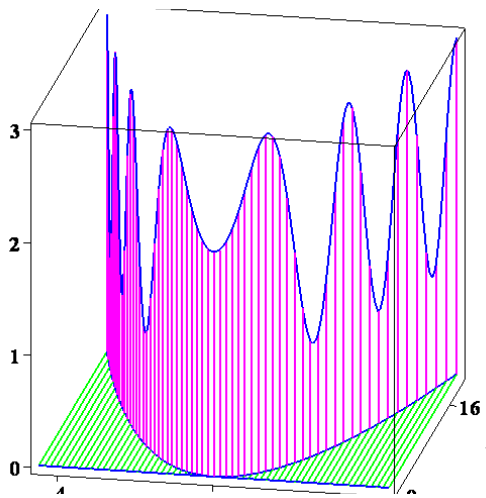


Fig. 3.c: The full 3D view

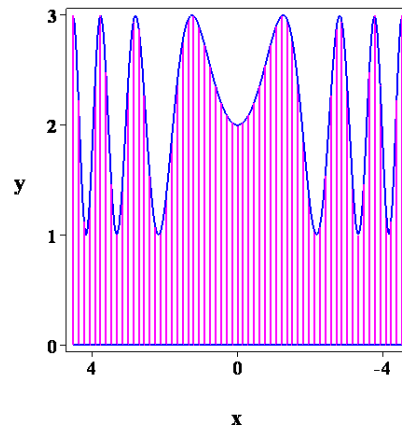


Fig. 3.d: The frontal view: $y = f(g(x))$

Denoting the composition by $F = f \circ g$, one has several options: The space-curve $t \rightarrow (t, g(t), F(t))$ is difficult to *see* (in some cases a point-plot is more helpful). The graphs of the equations $u = g(x)$ and $f = g(u)$ maybe interpreted as cylinders with the space-curve as their intersection. In our experience, the resulting surface plots are often unsatisfactory. Plotting only a judiciously chosen truncation of either surface is much more informative. Here we chose to include (a minor modification of) the “*lines of reasoning*” described in the hands-on illustration using card-board.

As so often in 3D graphics, here too, oscillatory trigonometric and bounded rational functions are more suitable than polynomial functions. However, due to popular demand, we illustrate the composition of $g(x) = x^2$ and $f(u) = 2 + \sin u$. The most delicate choice is the restriction of the domain to obtain just enough oscillations. Similar to the case of $\sin e^x$ (left to the reader), restricting x so that no more than three full periods of the sine are shown seems to give the most gratifying pictures.

As before, the side and the top view in Figures 3.a and 3.b show the basic graphs of $y = f(u)$ and $u = g(x)$. Then rotate into full 3D view as in Figure 3.c. Continue to slowly rotate until the u -axis disappears from view, corresponding to elimination of u from the two equations, compare Figure 3.d. The desired graph of $y = F(x)$ remains. The most memorable view keeps the u -direction just barely visible. In MAPLE it seems best to always use the “*BOXED*”-option for the axes.

Extension

Closely related to this picture are the chain-rule and the substitution-technique for integration. For the first, connect two points on the space curve by a secant line, and adjoin line segments parallel to the coordinate axis. Or draw a right three-sided pyramid with two vertices on the space-curve and three sides representing Δx , Δu , and Δv . The other sides approximate the arc lengths of the graphs of f , g and F .

Finally, one may illustrate the Riemann sums corresponding to the definite integrals transformed into each other by a change of variable. This picture in three dimensions takes a little more effort than the previous ones: For an integrand $J(x)dx = f(g(x))g'(x)dx$ use three axes to represent x , $u' = g'(x)$ (!) and $y = F(x) = f(g(x))$. Considering volumes of rectangular boxes (instead of areas of rectangles) we may group volume = (width \times depth) \times height = width \times (depth \times height). Here

$\Delta u \approx g'(x) \cdot \Delta x$ and $J(x) = f(g(x)) \cdot g'(x)$ are represented by the areas of different rectangular sides of the individual boxes.

5 Conclusion

For millennia mathematicians have gotten by without extensive 3D graphics. Yet there is every reason to believe that future users of mathematics can do much more by exploiting 3D visualization. There are compelling reasons to consider this approach as more than a entertaining diversion that keeps a few students more excited by being new, beautiful, and fun:

Three dimensional visualizations

- provide more insight,
- link together different views and techniques,
- practice the essential mathematical techniques of projecting and lifting,
- demonstrate abstractions and finding new illuminating views as central features of mathematics.

We hope that these and similar graphics eventually make it into standard textbooks. Until then, instructors can still do magic and present *new* ways of looking at mathematics. This may leave, on many students, a lasting impression that new, beautiful faces of mathematics are yet to be discovered, even at the calculus level! Many similar 3D examples need to be explored and many completely new uses of technology for ancient topics are waiting to be discovered.

Acknowledgments: The author gratefully acknowledges the feedback from his students and the encouraging comments by Robin Carr and Gilbert Strang. The author also thanks the anonymous referees for many detailed suggestions that improved the manuscript and the MAPLE code.

6 Supplementary Electronic Materials

Maple code

References

- [1] T. Dray and C. Manogue, *Putting differentials back into differential (and integral) calculus*, College Math. J. **41** (2010) pp. 90–100 (2010).
- [2] D. Hughes-Hallet, A. Gleason, W. McCallum, et al., *Calculus: Single and Multivariable*, 5th ed. (2009), Wiley.
- [3] P. Kosmol and M. Pavon, *Lagrange Lemma and Optimal Control of Diffusions: 1. Differentiable multipliers*, Proc. 31st IEEE Conf. Decision and Control (1992) pp. 2037 – 2042.

[4] D. Lomen and D. Lovelock, *Exploring Differential Equations via Graphics and Data*, (1996) Wiley.

[5] G. Strang, *Calculus* (1991) Wellesley-Cambridge Press.

7 Appendix

Graphics similar to the ones presented may be generated with almost any graphing package. Here we give sample *MAPLE* commands that may be used to generate illustrations similar to the ones presented. More elaborate commands are available in the *MAPLE* worksheet. These commands assume *MAPLE* release 13, but with only minor changes will also work in most recent versions.

Constrained optimization: The color essentially codes the fourth variable related to the perimeter constraint – the contours of the constraint function are represented by colors.

```
with(plots):
f:=(x,y)->x*y;
h:=x->6-x;
F:=x->f(x,h(x));
ss:=spacecurve([x,h(x),F(x)],x=0..6,color=blue,thickness=3):
xe:=solve(diff(F(x),x)=0):
This simplified assignment assumes that there is a unique critical point.
yc:=solve(f(x,y)=F(xe),y):
crit:=spacecurve([x,yc,F(xe)],x=3/2..6,color=red,thickness=3):
surf:=plot3d(f(x,y),x=0..6,y=0..6,color=h(x)-y,
style=patchcontour,contours=[\$1..12]*~3):
display({crit,ss,surf},labels=[x,y,A],
axes=boxed,orientation=[-90,0]);
display({crit,ss,surf},labels=[x,y,A],
axes=boxed,orientation=[0,90]);
display({crit,ss,surf},labels=[x,y,A],
axes=boxed,orientation=[-70,70]);
```

Product rule: The functions u and v for width and length may be changed easily. The picture consists of two rectangles, three space curves, and numerous line segments as visual aids. The space between the rectangles may be easily varied.

```
with(plots):
u:=x->1/(2-x):
v:=x->x*(1+x)/2:
R:=(x,cc)->polygonplot3d([[x,0,0],[x,u(x),0],[x,u(x),v(x)],
[x,0,v(x)]],color=cc):
ss:=spacecurve({[x,u(x),v(x),x=0..1],
[x,0,v(x),x=0..1],
[x,u(x),0,x=0..1]
```

```

        [x, 0, 0, x=0..1]},
        color=pink, thickness=3):
hh:=a->pointplot3d( [[1,u(a),v(1)], [1,u(a),v(a)], [1,u(1),v(a)]],
        style=line, color=gold, thickness=2 ):hh:=a->display(
        pointplot3d([[1,u(a),0], [1,u(a),v(1)]],
        style=line, color=gold, thickness=2),
        pointplot3d([[1,0,v(a)], [1,u(1),v(a)]],
        style=line, color=gold, thickness=2)):
kk := a -> display(
        pointplot3d( [[a,u(a),v(a)], [1,u(a),v(a)]],
        style=line, color=cyan, thickness=2 ),
        pointplot3d( [[a,0,v(a)], [1,0,v(a)]],
        style=line, color=cyan, thickness=2 ),
        pointplot3d( [[a,u(a),0], [1,u(a),0]],
        style=line, color=cyan, thickness=2 ),
        pointplot3d( [[a,0,0], [1,0,0]],
        style=line, color=cyan, thickness=2 ) ):
prodrule:=(a,c1,c2)->display([R(a,c1),R(1,c2),ss,kk(a),hh(a)],
        axes=normal, labels=[x,u,v]):
display( prodrule(0.8,green,blue), orientation=[160,75] );
display( prodrule(0.8,green,blue), orientation=[-90,0] );
display( prodrule(0.8,green,blue), orientation=[-90,90] );

```

Composition: The tricky part is to select suitable plot ranges and a suitable viewing angle, as well as adding visual aids such as the truncated vertical cylinder. The reader is invited to experiment with other cylinders and other truncations.

```

with(plots):
g:=t->t^2;
f:=t->2+sin(t);
a:=sqrt(13*Pi/2);
N:=30:
ss:=spacecurve([t,g(t),f(g(t))],t=-a..a,
        color=red,numpoints=200,thickness=2):
hh:=xk->pointplot3d([[xk,0,0],[xk,g(xk),0]],
        color=green,style=line,thickness=2):
vv:=xk->pointplot3d([[xk,g(xk),0],[xk,g(xk),f(g(xk))]],
        style=line,thickness=2):
xx:=[-30..30]*~(a/N):
display([hh(xx)[],vv(xx)[],ss],labels=[x,y,z],
        axes=boxed,orientation=[0,90]);
display([hh(xx)[],vv(xx)[],ss],labels=[x,y,z],
        axes=boxed,orientation=[-90,90]);
display([hh(xx)[],vv(xx)[],ss],labels=[x,y,z],
        axes=boxed,orientation=[55,60]);

```